In this note I explain in detail the isomorphism $(G/N)/(K/N) \simeq G/K$ mentioned in Lecture 13

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normal subgroups

Let $K \subset G$

$N \subset G$

Assume that $N \subset K$.

<u>Claim 1</u> $N \subset K$ is normal

<u>proof.</u> follows from the fact that $N \subset G$ is normal

So, we can form a quotient K/N

this is a <u>group</u>

<u>Claim 2</u> K/N is a subgroup of G/N

proof. construct a map

$$\begin{array}{ccc} K/N & \xrightarrow{i} & G/N \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ gN & \longmapsto & gN \end{array}$$

it is a homomorphism (by definitions)

i is injective: enough to check that

$\ker i = \{N\}$, indeed if $i(gN) = N$

$$\Downarrow$$

$$gN = N$$

Claim 3 $K/N \overset{i}{\hookrightarrow} G/N$

normal

proof. take $kN \in K/N$, $gN \in G/N$

then $(gN)\cdot(kN)\cdot(gN)^{-1} =$

$= gkg^{-1}N \in K/N$ (use that $gkg^{-1} \in K$)

Thm ("third" isomorphism theorem)

$$(G/N)\big/(K/N) \simeq G/K$$

isomorphism of groups

proof

consider the map

$$\begin{array}{ccc} G/N & \xrightarrow{\pi} & G/K \\ \cup & & \cup \\ gN & \longmapsto & gK \end{array}$$

1) π is well-defined: if we replace g by gn for some $n \in K$, then $gnK = gK$

use that $n \in N \subset K$

2) $\pi \leftarrow$ homomorphism of groups:

$$\pi(g_1 N \cdot g_2 N) = \pi(g_1 g_2 N) =$$

$$= g_1 g_2 K =$$

$$= g_1 K \cdot g_2 K = \pi(g_1 N) \cdot \pi(g_2 N) \checkmark$$

3) $\ker \pi = N/K$

indeed, $\pi(gN) = K \iff gK = K \iff$

$$\iff g \in K \iff gN \in N/K$$

4) $\pi \leftarrow$ surjective

preimage of $gK \in G/K$ is gN

So, we have a surjective homomorphism

$$\pi : G/N \twoheadrightarrow G/K$$

with kernel K/N.

By the general theorem we proved

$$G/K \simeq (G/N)/(K/N)$$

as desired

We use the following thm that we proved in Lecture 9

<u>Thm</u> if $S \twoheadrightarrow S'$ surjective homomorphism with kernel P, then

$$\underline{S' \simeq S/P}$$

$$\varphi$$

we apply this thm. to

$$S = G/N$$

$$S' = G/K$$

$$P = K/N$$